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SOME RESULTS ON MULTIPLICITIES FOR SL(n)

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ABSTRACT

We define the multiplicity and the global multiplicity of an L-packet of SL(n), unifying lack of multiplicity one and non-rigidity of L-packets. The first examples of these phenomena were given by Blasius. Giving a heuristic approach to its calculation, based on Langlands' Tannakian formalism, we conjecture that the global multiplicity is bounded in terms of n only. We justify the heuristics in a special case of L-packets attached to Hecke characters on an Abelian or p-extension. We then focus on L-packets lifted from endoscopic tori. A full description of their global multiplicities is given in the case where n is prime.

1. Introduction

Let G be a reductive group defined over a number field F. The cuspidal spectrum of $G(F)\backslash G(\mathbb{A}_F)$ (with a given central character) decomposes discretely into a sum of irreducible representations, each occurring with a finite multiplicity. In the case where $G = \mathrm{GL}(n)$ all multiplicities are one in this decomposition ([Sh]). Moreover, two cuspidal representations with the same Hecke eigenvalues almost everywhere are equivalent ([JS]). Cuspidal representations of SL(n) are intimately related to those of GL(n). However the situation for SL(n) changes dramatically. For example, it is known since [LL] that L-packets of SL(2) can be infinite, at least in the unstable case, hence naive strong multiplicity one cannot hold.

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More recently, Blasius constructed cuspidal representations of $\mathrm{SL}(n)$ of Galois type with multiplicity $\geq \varphi(n)$ (where φ is Euler's function). He also showed that strong multiplicity one does not hold in the level of L-packets, so that two representations which are a.e. the same do not have to belong to the same L-packet. In this paper we will be interested in these two phenomena which tie up in the definition of **global multiplicity** (see below). The high multiplicities for $\mathrm{SL}(n)$ are not surprising, since the cuspidal spectrum of $\mathrm{SL}(n)$ has a natural action of $\mathrm{GL}(n, F)$ on it by conjugation. If one takes into account those additional symmetries then the multiplicity is one. This is because

$$\mathrm{Ind}_{\mathrm{SL}_n(\mathbb{A})\mathrm{GL}_n(F)}^{\mathrm{GL}_n(\mathbb{A})} L^2_{cusp}(\mathrm{SL}_n(F)\backslash \mathrm{SL}_n(\mathbb{A})) = L^2_{cusp}(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}))$$

(cf. [LL]). From a different point of view, high multiplicity is related to the fact that two non-equivalent projective representations of a group may become equivalent when restricted to any cyclic subgroup. After defining the global multiplicity of an L-packet and giving some heuristics and examples, we will focus on a particularly handy case of L-packets which are liftings of endoscopic tori. The examples given in [B] are a special case of this. It turns out that in this case the global multiplicity is given naturally by an order of an Abelian group. The problem of computing the global multiplicity reduces to a completely algebraic question in representation theory of finite groups. Our results are most complete in the case where n is prime. In that case we can give a complete classification of the global multiplicities of endoscopic representations. To state our result, let $F \subset E$ be a cyclic extension of prime order p, and let E^1 be the torus of norm one elements. The Galois group $\mathbb{Z}_p$* and hence also the group ring $\mathbb{Z}[\mathbb{Z}_p]$, act on the Hecke characters of E^1. Let θ be a non-trivial Hecke character of E^1 and let $\mathrm{Ann}(\theta)$ be its annihilator in $\mathbb{Z}[\mathbb{Z}_p]$. The global multiplicity of the L-packet of $\mathrm{SL}_p(F)$ associated to θ is then given by the order of a group $G_\theta \subset \mathbb{Z}_p$ * depending only on $\mathrm{Ann}(\theta)$. The non-trivial G_θ are classified according to the following table

p	$\mathrm{Ann}(\theta)$	G_θ	$\lvert G_\theta\rvert$	Comments
any	$(\sigma - 1)$	$\mathbb{Z}_p^*$	$p-1$	Blasius' example
any	$((\sigma-1)^2)$	$(\mathbb{Z}_p^*)^2$	$\frac{p-1}{2}$	
$4k+1$	$((\sigma-1)^3)$	$\{\pm 1\}$	2	
$\frac{q^{r^s}-1}{q^{r^{s-1}}-1}$	$(q(\sigma-1), pg(\sigma))$	$\left(q^{r^{s-1}}\right)$	r	q,r primes, $g(x) \in \mathbb{Z}_q[x]$ $g(x) \mid \Phi_p(x) = \frac{x^p-1}{x-1}$, irreducible (of degree r^s)

* In this note $\mathbb{Z}_n$ will always denote the cyclic group $\mathbb{Z}/n\mathbb{Z}$.

(primes which are Fermat or Mersenne numbers are a typical example for the last row). The essential tool in proving, and even stating, the results is the **base change** lift proved first by Arthur and Clozel ([AC]) (see also [Lab]). In the non-endoscopic case we can give examples of high multiplicity only if we assume the global Langlands conjecture (see below). I do not know of any example (even conjectural) which is non-endoscopic and not of Galois type.

The contents of this paper are as follows. In Section 2 we review basic results about L-packets of SL(n). We observe that the multiplicity of all cuspidal representations within an L-packet is constant and we define it to be the multiplicity of the L-packet. We then define the global multiplicity (denoted by $\mathcal{M}(\cdot)$) of an L-packet to be the sum of multiplicities of all L-packets which coincide with it almost everywhere. We can adapt the multiplicity formula of [LL] to the global multiplicity. We conjecture that the global multiplicity is finite and bounded in terms of n only. Indeed, there is every reason to believe that this should be true for any reductive group G.

Section 3 is mostly heuristic. By analogy with the multiplicity formula we consider homomorphisms of a group into a given Lie group G. Two homomorphisms are equivalent ($\sim$) if one is a conjugate of the other by an element of G. A weaker notion (written $\sim_w$) is that the images of each element are conjugate in G. The notion was introduced in [GW] (cf. [Lar]). The two notions are the same for $G = \mathrm{GL}(n, \mathbb{C})$ but not for $\mathrm{PGL}(n, \mathbb{C})$. The deviation of weak equivalence from equivalence is "responsible" for high global multiplicities. This is made precise by Arthur's multiplicity formula [AG], which reduces to the multiplicity formula of [LL] in the SL(n) case. (In other cases, e.g. the group of norm one elements of a quaternion algebra, the multiplicity formula has another ingredient which may contribute to high multiplicity.) At any rate, the above deviation can be quantified and it is bounded in terms of G only. The argument resembles the usual proof of the finiteness of the number of nilpotent orbits in a Lie group ([Ri]), together with a theorem of Jordan on linear groups. If one believes the Tannakian formalism of [L1] then this verifies the conjecture made in Section 2, simply by taking $G = \mathrm{PGL}(n, \mathbb{C})$ (the L-group of SL(n)). Still motivated by the Tannakian formalism we study the difference between the above two equivalence notions more closely. Given a projective representation α of a group A, we define the set

$$\mathcal{X}(\alpha) = \{\beta\colon A \longrightarrow \mathrm{PGL}(n, \mathbb{C})\colon \beta \sim_w \alpha\}/\sim .$$

The group $\mathrm{Aut}_w(\alpha)$ of those automorphisms ϕ of $A/\operatorname{Ker}\alpha$ such that $\alpha \circ \phi \sim_w \alpha$, acts (non-transitively in general) on $\mathcal{X}(\alpha)$. If α is a projective representation of the Langlands' group then by the multiplicity formula the global multiplicity of the L-packet attached to α is $|\mathcal{X}(\alpha)|$. Of course, even if we replace the highly speculative Langlands group by the Weil group, one does not know in general how to attach an L-packet $\mathcal{L}(\alpha)$ to a projective representation α. Even in cases where $\mathcal{L}(\alpha)$ is known to exist, it is not automatic that $\mathcal{M}(\mathcal{L}(\alpha)) = |\mathcal{X}(\alpha)|$. We finish the section by proving this equality for the special case where α is induced from a character on an extension field E for which either $F \subset E$ is Abelian, or the normal closure of E over F is a p-extension.

In Section 4 we give several examples. In the first one, π is attached to a Stone–von-Neumann representation of a Galois group isomorphic to a Heisenberg group of a $2m$-dimensional symplectic space V over $\mathbb{Z}_p$. (More generally, V can be an Abelian group with a perfect alternating pairing on it.) The case $m = 1$ was considered in [B]. The heuristics of Section 3 are applicable and $\mathcal{X}(\alpha)$ has the structure of the homogeneous space $\mathrm{GL}(2m, \mathbb{Z}_p)/Sp(2m, \mathbb{Z}_p)$ (here $n = p^m$). In particular $\mathcal{M}(\pi)$ is larger than any polynomial in n. Also, in this example, it is possible that the multiplicities of the L-packets composing the "L-bag" do not divide the global multiplicity. In particular they are not all the same necessarily, giving more motivation to the definition of global multiplicity. The second is a non-solvable example due to Borovic [Bo]. α is now a 9-dimensional projective representation of A_6, the alternating group on 6 letters. This illustrates a case where $\mathcal{X}(\alpha)$ is not a homogeneous space. It is composed of two orbits of sizes 1 and 2. Evidently, we can build a Galois representation factoring through α. Unfortunately, with the present knowledge this example cannot be translated to the automorphic side. However, we also give another example for which $\mathcal{X}(\alpha)$ is non-homogeneous, and for which we can prove that the global multiplicity of the corresponding cuspidal L-packet is computed as $|\mathcal{X}(\alpha)|$. Finally, it can happen that two representations (or rather their projectivizations) are weakly equivalent, even though one is induced from a character on a subnormal subgroup and the other is not monomial, and the group is solvable. This illustrates the difficulties in generalizing Theorem 2.

The next three sections are devoted to a study of global multiplicities of the special case of L-packets defined by taking automorphic induction $\pi(\theta)$ of a Hecke character of a cyclic extension E of order n. These are the endoscopic L-packets corresponding to elliptic tori. In this case the above heuristics are applicable. Moreover, $\mathcal{X}(\alpha)$ turns out to be a homogeneous space which is actually an Abelian group which we denote by G_θ. Here α is the projectivization of the Weil group representation $\mathrm{Ind}_{W_E}^{W_F} \theta$. Thus the global multiplicity of the L-packet defined by

$\pi(\theta)$ is given by the size the group G_θ, which is fairly computable. As mentioned above, in the case where n is prime we have $G_\theta \leq \mathbb{Z}_n^*$ and we can actually classify G_θ in terms of the annihilator of θ in the group ring $\mathbb{Z}[\mathrm{Gal}(E/F)]$. This is done in Section 6. If n is a prime power, then we still have $|G_\theta|\,|\varphi(n)$. For general n we have $|G_\theta| < n$, but it is not true in general that $|G_\theta|\,|\varphi(n)$; we give an example for $n = 3q$, q prime that $G_\theta \simeq \mathbb{Z}_q$.

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2. L-packets in SL(n)

Let us recall the definition and the basic properties of L-packets in $G = \mathrm{SL}(n)$. See [LL], [B]. Recall that an irreducible representation π of $G(\mathbb{A})$ is called cuspidal, if $m(\pi) = \dim \mathrm{Hom}(\pi, L^2_{cusp}(G(F)\backslash G(\mathbb{A}))) > 0$.

Definition 1: Let $\tilde{\pi}$ be an equivalence class of an irreducible admissible representation of $\tilde{G} = \mathrm{GL}(n, \mathbb{A})$. The L-packet defined by $\tilde{\pi}$ (denoted by $\mathcal{L}(\tilde{\pi})$) is the set of equivalence classes of irreducible components of $\tilde{\pi}\big|_{G(\mathbb{A})}$.

2.1 REMARKS.

1. Equivalently, we can define an L-packet to be the orbit of an irreducible admissible representation of $G(\mathbb{A})$ under the natural action of $\tilde{G}(\mathbb{A})$.
2. There is an analogous definition of L-packets in the local case, and if $\tilde{\pi} = \otimes\tilde{\pi}^v$ then
$$\mathcal{L}(\tilde{\pi}) = \otimes\mathcal{L}(\tilde{\pi}^v) = \{\otimes\pi^v \colon \pi^v \in \mathcal{L}(\tilde{\pi}^v) \text{ for all } v, \pi^v \text{ unramified a.e.}\}.$$
3. Locally, if $\tilde{\pi}$ is generic then any $\pi \in \mathcal{L}(\tilde{\pi})$ is generic with respect to *some* non-degenerate character ψ of the maximal unipotent. If $\pi_1, \pi_2 \in \mathcal{L}(\tilde{\pi})$ are ψ-generic then $\pi_1 = \pi_2$. Thus $\tilde{\pi}\big|_G$ decomposes to a direct sum of pairwise inequivalent irreducible representations.
4. Although we shall not use this fact, let us note that even in the non-generic case, the decomposition of $\tilde{\pi}\big|_G$ is multiplicity free. The argument in [LL] still applies, except that one has to use existence and uniqueness of (possibly degenerate) Whittaker models (see [Z]). (For another approach using Langlands' classification see [T].)

5. $\mathcal{L}(\tilde{\pi}_1) = \mathcal{L}(\tilde{\pi}_2)$ if and only if there exists an admissible character ω such that $\tilde{\pi}_2 \simeq \tilde{\pi}_1 \otimes \omega$ (locally and globally).

6. If $\tilde{\pi}$ is cuspidal then $\mathcal{L}(\tilde{\pi})$ contains a cuspidal representation. Conversely, any cuspidal π belongs to an L-packet which can defined by a cuspidal representation of $\tilde{G}$.

7. An L-packet is called **stable** if all representations in it appear with the same multiplicity in the cuspidal spectrum. It is conjectured (see [L4]) that an unstable L-packet is **endoscopic**, i.e. it can be defined by $\tilde{\pi}$ which is an **automorphic induction** from a cyclic extension $F \subset E$ ([AC]) (or equivalently, $\tilde{\pi} \otimes \omega \simeq \tilde{\pi}$ for some Hecke character $\omega \neq 1$). This would be a consequence of the stable trace formula for G, as was done in the case of SL(2) in [LL].

8. Even in the unstable case, it is still true that all **cuspidal** representations in an L-packet $\mathcal{L}$ have the same multiplicity in the cuspidal spectrum. Indeed, if $\pi_1, \pi_2 \in \mathcal{L}$ are cuspidal (or even generic), we can find an element $\tilde{g} \in \tilde{G}(F)$ such that $\pi_1, \pi_2^{\tilde{g}}$ are generic with respect to the same character ψ of $\mathbb{A}_F/F$. Consequently, by 3, $\pi_2 \simeq \pi_1^{\tilde{g}}$ and hence $m(\pi_2) = m(\pi_1)$. Thus it makes sense to speak about **the** multiplicity of an L-packet, and we'll denote it by $M(\mathcal{L})$.

9. Let us define equivalence relations on cuspidal representations of $\tilde{G}$:

   (a) $\tilde{\pi}_1 \sim_s \tilde{\pi}_2$ if there exists a Hecke character ω of $C_F = \mathbb{I}_F/F^*$ such that $\tilde{\pi}_2 \simeq \tilde{\pi}_1 \otimes \omega$.

   (b) $\tilde{\pi}_1 \sim_{ew} \tilde{\pi}_2$ if for each place v there exists a character ω^v of F_v^* such that $\tilde{\pi}_2^v \simeq \tilde{\pi}_1^v \otimes \omega^v$.

   (c) $\tilde{\pi}_1 \sim_w \tilde{\pi}_2$ if for almost all v the above holds.

   The multiplicity formula (which is written in [LL] in a somewhat confusing way) says that for a cuspidal $\tilde{\pi}$, $M(\mathcal{L}(\tilde{\pi}))$ can be calculated as the number of $\sim_s$-classes in the $\sim_{ew}$-class of $\tilde{\pi}$. In particular this number is finite.

Let us call two L-packets **inseparable** (denoted by $\doteq$) if locally they are the same almost everywhere. Let us define the **global multiplicity** as $\mathcal{M}(\mathcal{L}) = \sum_{\mathcal{L}' \doteq \mathcal{L}} M(\mathcal{L}')$. Thus $\mathcal{M}(\mathcal{L}(\tilde{\pi}))$ is given by the number of $\sim_s$-classes in the $\sim_w$-class of $\tilde{\pi}$. In general, it is not clear why this is always finite. We conjecture the following:

CONJECTURE 1: *There exists a constant $c(n)$ such that $\mathcal{M}(\mathcal{L}) \leq c(n)$ for every L-packet $\mathcal{L}$ of $G = \mathrm{SL}(n)$.*

We will show that this conjecture is at least in accordance with the Tannakian formalism of [L1]. An analogous conjecture should be true for any reductive group. For $n = 2$ it was shown in [LL] that the multiplicities of the unstable L-packets are 1; in the stable case Ramakrishnan ([Ra]) uses a nice L-function argument and a converse theorem for GL(4) to show that, in the above language, global multiplicities are 1. Thus we can take $c(2) = 1$. For $n > 2$ Blasius gave examples where $M(\mathcal{L}) > 1$, as well as examples for which $\mathcal{L}_1 \doteq \mathcal{L}_2$ but $\mathcal{L}_1 \neq \mathcal{L}_2$. Actually, both phenomena are implemented by the same type of examples, and we'll see that for these examples $\mathcal{M}(\mathcal{L}) = \varphi(n)$.

3. Some heuristics

Let G be a reductive group over $\mathbb{C}$ and ϕ: $G \longrightarrow \mathrm{GL}(N, \mathbb{C})$ a faithful representation of it. Let H be a topological group. Consider continuous homomorphisms π: $H \longrightarrow G$ — call them G-representations. Two G-representations π_1, π_2 are called G-equivalent (denoted by $\pi_1 \sim_G \pi_2$) if there exists $g \in G$ such that $\pi_1(h) = g^{-1}\pi_2(h)g$ for any $h \in H$. They are $\sim_\phi$-equivalent if after composition with ϕ they become equivalent representations (in the usual sense). Finally $\pi_2 \sim_w \pi_1$ if for any element $a \in H$, $\pi_1(a)$ and $\pi_2(a)$ are conjugate in G (cf. [Lar], [GW]). When there is no ambiguity about the group G we will sometime write $\sim_s$ for $\sim_G$. It is clear that $\sim_s \Rightarrow \sim_w \Rightarrow \sim_\phi$ for any ϕ.

THEOREM 1: *Let G, ϕ be as above. There exists a constant C, depending only on G and N, such that for any compact group H and any G-representation π of it, the number of $\sim_s$-classes inside the $\sim_\phi$-class of π is bounded by C.*

Proof: The proof will proceed in the following steps.

Note: Henceforth we shall denote by $c_i(D)$ constants which depend only on the data D.

1. If $H_1 < H$ is of finite index k then a G-representation of H_1 can be extended to H in only finitely many ways up to G-equivalence. This number is bounded by $c_1(k, G)$.

   The proof is a straightforward generalization of that of Theorem 3.1 in [Ri] (cf. [Bo]). We want to reduce the claim to the case $G = \mathrm{GL}(N, \mathbb{C})$, where it is well known. Let π be a G-representation of H_1 and choose a transversal Γ of H_1 in H. We will identify G, through ϕ, with a subgroup of $G_1 = \mathrm{GL}(V)$. Let $\mathfrak{g}, \mathfrak{g}_1$ denote the corresponding Lie algebras. Let U be a complement of $\mathfrak{g}$ in $\mathfrak{g}_1$ which is $\mathrm{Ad}_{\mathfrak{g}_1} g$-invariant for any $g \in G$. We shall regard extensions of π to G (resp. G_1)-representations of H as points in $X = G^\Gamma$ (resp. $X_1 = G_1^\Gamma$) simply by specifying them on Γ. G acts on X by conjugation in each component – and analogously G_1 acts on X_1. Two extensions of π to representations of H become equivalent in G (resp. G_1) if and only if the corresponding points lie in the same orbit of $G' = C_G(\pi(H_1))$ (resp. $G'_1 = C_{G_1}(\pi(H_1))$). Thus we need to prove that a G'_1-orbit O_1 in X_1 intersects only a bounded number of G'-orbits in X. We will prove that any G'-orbit is open in $O_1 \cap X$. For this we work infinitesimally. Let $\mathfrak{g}', \mathfrak{g}'_1$ be the Lie algebras of G', G'_1 respectively. Regard X_1 as a subvariety of $\mathrm{End}(V)^\Gamma$. Denote by $T(Y, y)$ the tangent space of a variety Y at a point y, and identify $T(Y, y)$ with a subspace of $T(Y_1, y)$ if $Y \subset Y_1$. Take $x \in W = O_1 \cap X$. Then $T(O_1, x) = [\mathfrak{g}'_1, x]$ (here by an abuse of notations, the bracket is taken in $\mathfrak{gl}(V)$ coordinatewise). Also, $T(X, x) = x\mathfrak{g}^\Gamma$ where the product is taken in $\mathrm{End}(V)$ coordinatewise. Let Z be an irreducible component of W containing x. Then $T(Z, x) \subset T(O_1, x) \cap T(X, x) = [\mathfrak{g}'_1, x] \cap x\mathfrak{g}^\Gamma$. Let $z \in T(Z, x)$. Then there exist $y \in \mathfrak{g}'_1, g \in \mathfrak{g}^\Gamma$ such that $z = [y, x] = xg$. Thus in each component $(\mathrm{Ad}_{\mathfrak{g}_1} x_\gamma)(y + g_\gamma) = y$. Writing $y = y_0 + y_1$ with $y_0 \in \mathfrak{g}$ and $y_1 \in U$ we find that necessarily $y_0 = (\mathrm{Ad}_{\mathfrak{g}_1} x_\gamma)(y_0 + g_\gamma)$. Moreover, since $y \in \mathfrak{g}'_1$: $y_0 + y_1 = y = (\mathrm{Ad}_{\mathfrak{g}_1} \pi(h))y = (\mathrm{Ad}_{\mathfrak{g}_1} \pi(h))(y_0) + (\mathrm{Ad}_{\mathfrak{g}_1} \pi(h))(y_1)$ for any $h \in H_1$. We infer that $y_0 \in \mathfrak{g}'$ and $z = [y_0, x] \in [\mathfrak{g}', x] = T(O, x)$ where O is the G'-orbit of x. Thus, we get that O is open in O_1. Since this is true for any O there are only finitely many G'-orbits in W, say $m(O_1)$. To bound this uniformly, we know that $m(O_1)$ is bounded by the number of connected components of $O_1 \cap X$. This, in turn, is majorized by the number of connected components of the fibers of the map $G'_1 \times X \longrightarrow X_1$ given by the action. This is bounded by a general theorem in algebraic geometry. We still have a-priori dependence on G'_1. However G'_1 can have only $\leq c_2(G)$ possibilities up to conjugacy in G_1. In each conjugacy class we get the required boundedness by considering fibers of the map $G_1 \times G'_1 \times X \longrightarrow G_1 \times X_1$ defined by $(g_1, g, x) \mapsto (g_1, g^{g_1} \cdot x)$.

2. The Theorem is true in the case where $H = \mathbb{T}$ or $\mathbb{Z}_n$.

By step 1 we can assume that G is connected. Let T be a maximal torus in G. Since H contains a dense cyclic subgroup, the image of H can be conjugated into T. Thus we are reduced to the case where G is a torus, where the claim is easy.

3. Any compact connected semisimple Lie group admits only finitely many G-representations up to G-equivalence.

   This is true for $G = \mathrm{GL}(n)$ and the general case follows from Theorem 7.1 in [Ri].

4. Let $H < \mathrm{GL}(n, \mathbb{C})$ be compact and assume that $H/Z(H)$ is finite. Then H contains an Abelian normal subgroup of index $\leq c_3(n)$.

   We can assume that H acts irreducibly. Then $\bar{H} = H/Z(H)$ can be embedded in $\mathrm{GL}(n^2, \mathbb{C})$ and hence by Jordan's theorem it contains an Abelian normal subgroup of index $\leq c_4(n)$ ([CR]). Thus we can assume that $\bar{H}$ is Abelian. In particular H is nilpotent. Still assuming, as we may, that H acts irreducibly and faithfully, the action is induced from some character on some subgroup H_0. Hence any normal subgroup $K < H_0$ is Abelian.

5. Let $H < \mathrm{GL}(n, \mathbb{C})$ be compact. Then H contains a subgroup K of index $\leq c_5(n)$ of the form $H_0 A$ where H_0 is a connected semisimple compact group and $A < Z(K)$.

   Let H_1 be the connected component of H. There are only finitely many compact connected subgroups in $\mathrm{GL}(n, \mathbb{C})$ up to isomorphism. Write $H_1 = H_1' \cdot Z(H_1)$ where H_1' is the derived group of H_1. The canonical homomorphism ψ: $H/H_1 \longrightarrow \mathrm{Out}(H_1) = \mathrm{Aut}(H_1)/\mathrm{Inn}(H_1)$ factors through a quotient of size $\leq c_6(n)$. This is because $\mathrm{Out}(H_1)$ embeds in $\mathrm{Aut}(Z(H_1)) \times \mathrm{Out}(H_1') \simeq \mathrm{GL}(r, \mathbb{Z}) \times \mathrm{Out}(H_1')$ and it is a classical result that finite subgroups of $\mathrm{GL}(r, \mathbb{Z})$ have bounded order (see e.g. [S1]). Let $H_2 = \mathrm{Ker}\,\psi$. Then $H_2 = H_1 C$ where $C = C_H(H_1)$. C satisfies the conditions of 4 and hence contains an Abelian subgroup C_0 as above. Clearly $K = H_1' C_0$ satisfies the desired property.

6. Finally, we can prove Theorem 1. Let π be a G-representation of H, which we can assume to be faithful. By step 1 we can suppose that $H = H_0 A$ as in 5. Let M be the centralizer of $\pi(H_0)$ in G, $\pi_A = \pi\big|_A$: $A \longrightarrow M$ and $\phi_M = \phi\big|_M$. Note that there are only finitely many possibilities for M up to isomorphism. Any other G-representation π' of H with $\pi' \sim_\phi \pi$ which is G-equivalent to π on H_0 gives rise to an M-representation of A, denoted by π_A', with $\pi_A' \sim_{\phi_M} \pi_A$. Clearly, if $\pi_A' \sim_M \pi_A$ then $\pi' \sim_G \pi$, so using step 3 we are reduced to the case where H is Abelian. H is then

the closure of a subgroup generated by d elements with $d \leq c_7(G)$. Let $H = \overline{\langle x_1 \rangle} \times H_1$. By using step 2 and the same argument as before we are reduced to the same question about M-representations of H_1 where $M = C_G(\pi(x_1))$. Thus we can use induction on d to get the required. ∎

To see how to apply the theorem to study multiplicities, assume that the Tannakian formalism of [L1] exists. Recall that this formalism implies the existence of the so-called Langlands group $\mathcal{L}_F$ whose irreducible n-dimensional representations correspond to the cuspidal representations of $\mathrm{GL}(n)$. As in [Ko] §12 we work with a form of it which is a compact group times $\mathbb{R}$. Thus the image of the projectivization of any irreducible representation of $\mathcal{L}_F$ is compact. Let $G' = \mathrm{PGL}(n, \mathbb{C})$ (the L-group of $G = \mathrm{SL}(n)$), and $\mathrm{Ad}\colon G' \longrightarrow \mathrm{GL}(n^2, \mathbb{C})$ be the adjoint representation. From now on $\sim_s$ will stand for $\sim_{G'}$. Let ψ be an irreducible n-dimensional representation of $\mathcal{L}_F$, and let $\mathcal{L}_\psi$ be the corresponding L-packet in G. This L-packet corresponds to the projectivization $\bar{\psi}$ of ψ into G'. Suppose that the cuspidal representations $\pi_i, i = 1, 2$ correspond to the irreducible n-dimensional representations ψ_i of $\mathcal{L}_F$. Then in the notations of 2.1.9, $\pi_1 \sim_s \pi_2$ if and only if $\bar{\psi}_1 \sim_s \bar{\psi}_2$ and the same for $\sim_w$. Thus, the multiplicity formula implies that $\mathcal{M}(\mathcal{L}_\psi)$ is majorized by the the number of $\sim_s$-classes in the $\sim_{ad}$-class of $\bar{\psi}$. Hence Conjecture 1 is compatible with the Tannakian formalism in view of Theorem 1. To obtain the conjecture for a general reductive group G, we also need to know that the degrees of the virtual characters appearing in Arthur's multiplicity formula are bounded in terms of G only.

Still motivated by the Tannakian formalism, let α be a projective representation α of a group A. We define the set

$$\mathcal{X}(\alpha) = \{\beta\colon \beta \sim_w \alpha\} / \sim_s$$

measuring the difference between $\sim_s$ and $\sim_w$. Let $\mathcal{M}(\alpha) = |\mathcal{X}(\alpha)|$. If α is a projective representation of $\mathcal{L}_F$ corresponding to the L-packet $\mathcal{L}$ then one expects by the multiplicity formula that $\mathcal{M}(\mathcal{L}) = \mathcal{M}(\alpha)$. We will be able to prove this relation in a special case below, with $\mathcal{L}_F$ replaced by the Weil group W_F. The set $\mathcal{X}(\alpha)$ has an additional structure. Let $\mathrm{Aut}_w(\alpha)$ be the group of those automorphisms ϕ of $A/\operatorname{Ker}\alpha$ such that $\alpha \circ \phi \sim_w \alpha$. Then $\mathrm{Aut}_w(\alpha)$ acts on $\mathcal{X}(\alpha)$. This action is not transitive in general. In other words, it may happen that $\beta \sim_w \alpha$ but their image subgroups in $\mathrm{PGL}(n, \mathbb{C})$ are not conjugate. The stabilizer of α under the action is $\mathrm{Aut}_s(\alpha) = \{\phi \in \mathrm{Aut}(A/\operatorname{Ker}\alpha)\colon \alpha \circ \phi \sim_s \alpha\}$.

We cannot say much in general about the structure of the set $\mathcal{X}(\alpha)$. See examples in the next section.

Let us introduce some more terminology and notations. We will deal with automorphic representations **induced from cuspidals**, in the language of [AC]. This class is preserved under cyclic base change and automorphic induction. We write these operations as BC_F^E and AI_E^F respectively. These operations are known to exist also if $F \subset E$ is solvable, or more generally, if there exists a series of cyclic extensions from F to E. If π_i are cuspidal representations of $\mathrm{GL}(n_i, \mathbb{A}_F)$ we shall write $\pi = \boxplus \pi_i$ for the representation parabolically induced from the π_i's. By abuse of language we call π the direct sum of the π_i's. We also call π_i the components of π. If $\pi = \boxplus \pi_i, \pi' = \boxplus \pi'_j$ let $c(\pi, \pi') = \#\{(i,j)\colon \pi_i \simeq \pi'_j\}$. This is the same as the order of the pole at $s = 1$ of the (partial) Jacquet–Shalika L-function $L(\tilde{\pi} \otimes \pi', s)$. We have the following Frobenius reciprocity:

$$c(\pi, \mathrm{AI}_E^F(\pi')) = c(\mathrm{BC}_F^E(\pi), \pi') \tag{1}$$

whenever π, π' are on $\mathrm{GL}(n, \mathbb{A}_F)$ and $\mathrm{GL}(m, \mathbb{A}_E)$ respectively. For any two cuspidal representations π, π' of $\mathrm{GL}(n, \mathbb{A}_F)$ we define

$$X(\pi, \pi') = \{\omega \in \widehat{C_F}\colon \pi \otimes \omega \simeq \pi'\}$$

and we put $X(\pi) = X(\pi, \pi)$. Let us call a Weil group representation σ automorphic if there exists an automorphic representation $\tilde{\pi}(\sigma)$ of $\mathrm{GL}(n)$, necessarily unique, with matching Langlands parameters almost everywhere. (Of course, conjecturely, every Weil group representation is automorphic.) We say that $\tilde{\pi}(\sigma)$ is of Galois type. Finally we call an extension $F \subset E$ p-subnormal if it can be embedded in a normal extension of F of p-power order. Recall that by [AC] we know that every representation induced from a Hecke character of a p-subnormal (or even sub-nilpotent) extension is automorphic.

LEMMA 1:

1. *Let π be a cuspidal representation of $\mathrm{GL}(n, \mathbb{A}_F)$ and assume that $|X(\pi)| \geq n^2$. Then π is of Galois type, $|X(\pi)| = n^2$, and the base change of π to the Abelian extension of F defined by $X(\pi)$ is equivalent to n times a Hecke character.*
2. *Let $F \subset E$ be a solvable Galois extension and π be a cuspidal representation of $\mathrm{GL}(n, \mathbb{A}_F)$. Suppose that ρ is a component of $\mathrm{BC}_F^E \pi$ and let $F \subset L \subset E$ be the field defined by $H = \{\sigma \in \mathrm{Gal}(E/F)\colon \rho^\sigma \simeq \rho\}$. Suppose that $F \subset L$ is subnormal. Then π is induced from L.*

Proof:

1. By Theorem 4.2 in ([AC]) we can write π as the automorphic induction of some cuspidal representation χ of $\mathrm{GL}(m, \mathbb{A}_E)$ with E/F cyclic and $m[E\colon F] = n$. We claim that $|X(\chi)| \geq m^2$. Indeed, $\pi \otimes \omega \simeq \pi$ implies that $\chi \otimes \omega_E \simeq \chi^\sigma$ for some $\sigma \in \mathrm{Gal}(E/F)$ where $\omega_E = \omega \circ \mathrm{Nm}_F^E$. Thus for some σ, $|X(\chi, \chi^\sigma)| \geq m^2$. But then, $|X(\chi)| = |X(\chi, \chi^\sigma)| \geq m^2$. By induction, π is induced from a Hecke character, and hence is of Galois type. The last two statements are true since they hold in the Galois side.

2. Let $\mathrm{BC}_F^E(\pi) = d \boxplus_{g \in H\backslash G} \rho^g$ and $\mathrm{BC}_F^L(\pi) = \boxplus d_i \rho_i$ with ρ_i cuspidal. Suppose without loss of generality that $\mathrm{BC}_L^E(\rho_1)$ contains ρ as a component. By the definition of L and the properties of base change we must have $\mathrm{BC}_L^E(\rho_1) = m\rho$ for some m. We claim that $\pi = \mathrm{AI}_L^F(\rho_1)$. Indeed, $c(\pi, \mathrm{AI}_L^F \rho_1) \geq 1$. However $\deg(\mathrm{AI}_L^F(\rho_1)) = [G\colon H] m \deg(\rho) \leq [G\colon H] d \deg(\rho) = n$, and we get the required. ∎

THEOREM 2: *Assume that $F \subset E$ is an extension of number fields which is either p-subnormal or Abelian. Let θ be a Hecke character of E and $\xi = \mathrm{Ind}_{W_E}^{W_F} \theta$. Suppose that $\tilde{\pi} = \tilde{\pi}(\xi)$ is cuspidal (i.e. ξ is irreducible). Then $\mathcal{M}(\mathcal{L}(\tilde{\pi})) = \mathcal{M}(\xi)$.**

Proof: Note that it will be enough to prove the following:

$$\text{Every } \tilde{\pi}' \sim_w \tilde{\pi} \text{ is of Galois type} \tag{2}$$

and

$$\text{Every } \xi' \sim_w \xi \text{ is automorphic.} \tag{3}$$

This will imply the Theorem, because the definitions of $\sim_s$ and $\sim_w$ in the automorphic and in the Galois side are compatible by Chebotarev's density theorem. Let then $\tilde{\pi}'$ be given. We prove (2) using the results of [AC] by the following steps.

1. Let K be the normal closure of $F \subset E$ and let τ, τ' be the base change to K of $\tilde{\pi}, \tilde{\pi}'$ respectively. By the properties of base change we have $\tau \sim_w \tau'$. Let $G = \mathrm{Gal}(K/F)$. By the properties of base change we can write $\tau = \boxplus_{g \in \mathrm{Gal}(K/E)\backslash G} \theta_K^g$ where $\theta_K = \theta \circ \mathrm{Nm}_E^K$, and $\tau' = d \boxplus_{H\backslash G} \rho^g$ where ρ is a cuspidal representation of some $\mathrm{GL}(m, \mathbb{A}_K)$ and $H = \{g \in G\colon \rho^g \simeq \rho\}$.

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* Recently, we were able to prove the Theorem for $F \subset E$ nilpotent (see [Lap2]).

2. We have $|X(\rho)| \geq m^2$.

   Indeed, since $\tau \sim_w \tau'$ we have an equality

$$L^S(\tau \otimes \hat{\tau} \otimes \omega, s) = L^S(\tau' \otimes \widehat{\tau'} \otimes \omega, s) \tag{4}$$

   of Jacquet–Shalika L-functions for any Hecke character ω of K. By the results of [JS]

$$\sum_\omega \mathrm{Ord}_{s=1} L^S(\tau \otimes \hat{\tau} \otimes \omega, s) = n^2,$$

   while

$$\sum_\omega \mathrm{Ord}_{s=1} L^S(\tau' \otimes \widehat{\tau'} \otimes \omega, s) = d^2 \sum_{g,g' \in H\backslash G} |X(\rho^g, \rho^{g'})|.$$

   Thus, $|X(\rho^g, \rho^{g'})| \geq m^2$ for some $g, g' \in H\backslash G$. Then necessarily $|X(\rho)| = |X(\rho^g)| = |X(\rho^g, \rho^{g'})| \geq m^2$.

3. By Lemma 1, ρ is of Galois type and

$$|X(\rho)| = m^2. \tag{5}$$

4. Let $K \subset L$ be the Abelian extension defined by $X(\rho)$. Then $F \subset L$ is Galois and the base change of π' to L is a sum of characters.

   By the argument of step 2, taking into account (5), we must have $|X(\rho, \rho^g)| = m^2 > 0$ for any $g \in H\backslash G$ and hence $X(\rho)^g = X(\rho^g) = X(\rho)$. The other statement follows from part 1 of Lemma 1.

5. Any component of a representation induced from a Hecke character of a nilpotent extension is of Galois type.

   Indeed, if $L \subset M$ is nilpotent and κ is a Hecke character of M then it is easy to see that we have a decomposition $\mathrm{Ind}_{W_M}^{W_L} \kappa \simeq \oplus \mathrm{Ind}_{W_{M_i}}^{W_L} \lambda_i$, into a sum of irreducible representations, with Hecke characters λ_i of fields $M \subset M_i \subset L$. Then $\tilde{\pi}(\kappa) \simeq \boxplus \tilde{\pi}(\lambda_i)$ is the decomposition of $\tilde{\pi}(\kappa)$.

6. Suppose that $F \subset K$ is a p-extension. Then by (1) and part 1 of Lemma 1, $\tilde{\pi}'$ is a component of a representation induced from a Hecke character of L. This proves (2) in that case.

7. Let us turn to the case where $F \subset E$ is Abelian. First, we claim that $d = m$.

For now, $E = K$ and we know that $\theta^g \neq \theta$ for any $1 \neq g \in G$. Comparing the orders of the poles at $s = 1$ for $\omega = 1$ in (4) we get $n = d^2[G\colon H]$. On the other hand $n = d[G\colon H]m$.

8. Let us write $c_g(\chi) = \chi^g/\chi$ for any $g \in G$ and a Hecke character χ of E. Define $H' = \{g \in G\colon c_g(\theta) \in X := X(\rho)\}$. Then H' is a subgroup of order d^2, and if $F \subset M' \subset E$ is the field defined by H' then $M' \subset L$ is nilpotent.

   Indeed, it is easy to see that H' is a subgroup. Also, from (4) we have an equality
   $$\{c_g(\theta)^{g'}\colon g, g' \in G\} = d^2 \bigcup_{g,g' \in H\backslash G} X(\rho^g, \rho^{g'})$$
   of multisets. Every element of X appears in the right hand side $d^2[G\colon H] = n$ times. This implies that $|H'| = d^2$. In other words, $g \mapsto c_g(\theta)$ is a bijection between H' and X. Let X_p be the p-Sylow subgroup of X. Then $H'_p := \{g \in G\colon c_g(\theta) \in X_p\}$ is the p-Sylow subgroup of H'. Suppose that $g \in H'_p$ and $g' \in H'_q$ for $p \neq q$. Then $c_g(c_{g'}(\theta)) = c_{g'}(c_g(\theta)) \in X_p \cap X_q = 0$. Thus, H'_p acts trivially on X_q. It follows that L/M' is nilpotent.

9. Let $F \subset M \subset E$ be the field defined by H. Then $M \subset L$ is nilpotent.

   Equivalently, there exists k so that $c_{h_1}(\dots c_{h_k}(x)\dots) = 1$ for any $h_1, \dots, h_k \in H$ and $x \in X$. We know that such a relation holds if $h_1, \dots, h_k \in H'$. Thus, it will be enough to show that for any $h_1, h_2 \in H$ and $x \in X$ there exists $h' \in H'$ and $x' \in X$ such that $c_{h_1}(c_{h_2}(x)) = c_{h'}(x')$. This will follow if we know that $c_h(c_g(\theta)) \in X$ for any $h \in H, g \in G$. To see this, let $\omega = c_g(\theta) \in X(\rho^{g_1}, \rho^{g_2})$ for some $g_1, g_2 \in G$. Then
   $$\rho^{g_1} \otimes \omega \simeq \rho^{g_2} = \rho^{g_2 h} \simeq \rho^{g_1 h} \otimes \omega^h = \rho^{g_1} \otimes \omega^h$$
   whence $c_h(\omega) \in X$ as required.

10. From 4,5 and part 1 of Lemma 1 (applied to $\tilde{\pi}'$) we derive (2) in the Abelian case.

11. The statement (3) is proved similarly. ∎

4. Some examples

We first begin with a definition.

Definition 2: Let G be a finite group with center Z and let $\bar{G} = G/Z$.

1. A faithful irreducible n-dimensional representation π of G (or the corresponding projective representation $\bar{\pi}$ of $\bar{G}$) is called **minimal** if the image of π in $\mathrm{PGL}(n, \mathbb{C})$ has order n^2. (Note that by Burnside's theorem, the above order is always $\geq n^2$.)

2. a cocycle $\alpha \in H^2(\bar{G}, \mathbb{C}^*)$ is called minimal if the twisted group algebra $\mathbb{C}[\bar{G}, \alpha]$ is simple.

LEMMA 2: *The following are equivalent for a faithful irreducible n-dimensional representation π of a finite group G.*

1. *π is minimal.*
2. *The cocycle on $\bar{G}$ defined by the projective representation $\bar{\pi}$ is minimal.*
3. *$\mathrm{Ind}_Z^G \zeta$ is isotypic where ζ is the central character of π.*
4. *The character of π vanishes outside Z.*

In that case, any irreducible n-dimensional representation π' of G is minimal and $\bar{\pi'} \sim_w \bar{\pi}$ (in $\mathrm{PGL}(n, \mathbb{C})$).

Proof: The equivalence of the above conditions is a standard exercise in representation theory of finite groups. The other assertion follows from 4. ∎

For W Abelian, the Schur multipliers correspond to alternating bilinear forms, and the minimal cocycles correspond to the non-degenerate ones. Minimal representations and cocycles were first studied (not under this name) in [IM] (see [S2] p. 172 for another appearance of them).

1. Our first example is an Abelian group W of order n^2 with a non-degenerate alternating form Q on it. This gives rise to an irreducible n-dimensional minimal projective representation α of W. Then $\mathrm{Aut}_w(\alpha) = \mathrm{Aut}(W)$. On the other hand $\mathrm{Aut}_s(\alpha) = \mathrm{Sp}(W, Q)$. Any other projective minimal representation of W is of this form, and since all non-degenerate alternating forms are conjugate, we conclude that $\mathcal{X}(\alpha) = \mathrm{Aut}(W)/\mathrm{Sp}(W, Q)$. In particular, if W be $2m$-dimensional vector space over $\mathbb{Z}_p$, α is the Stone–von-Neumann representation of the corresponding Heisenberg group and $\mathcal{X}(\alpha) = \mathrm{GL}(2m, \mathbb{Z}_p)/\mathrm{Sp}(2m, \mathbb{Z}_p)$. Of course, Theorem 2 is applicable here (and in fact, its proof in this case is much easier).

This example is also useful in showing that in general we don't have $M(\mathcal{L})|\mathcal{M}(\mathcal{L})$. In particular we can have $\mathcal{L} \doteq \mathcal{L}'$ with different (positive) multiplicities. Indeed, take $m = 2$ and take a 2-dimensional subspace V of W, generated by v_1, v_2. Let β: $\mathrm{Gal}(\bar{F}/F) \longrightarrow \mathrm{PGL}(n, \mathbb{C})$ be a projective Galois representation which implements α and let $\tilde{\beta}$: $\mathrm{Gal}(K/F) \longrightarrow \mathrm{GL}(n, \mathbb{C})$ be a lifting of it to a Galois representation. We can construct it so that K is ramified over F at exactly one place v in which $\beta(D_v) \sim \alpha(V)$ where D_v is the decomposition group. For any $\psi \in \mathrm{Aut}(W)$, let $\mathcal{L}_\psi$ be the L-packet attached to the projective Galois representation $\psi \circ \beta$ (that is, the one which is defined by a cuspidal representation corresponding to a Galois representation lifting it). In this case the correspondence of Galois representations to automorphic representations is functorial (globally and locally). We can infer by the multiplicity formula that $M(\mathcal{L}_\psi)$ is given by the number of ψ' in $\mathrm{GL}(4, \mathbb{Z}_p)/\mathrm{Sp}(4, \mathbb{Z}_p)$ such that $\psi'\big|V \sim_s \psi\big|V$. In other words, the condition is that $Q(\psi'(v_1), \psi'(v_2)) = Q(\psi(v_1), \psi(v_2))$. There are $(p^4 - 1)(p^3 - p)(p^4 - p^2)(p^4 - p^3)$ such ψ'-s (in $\mathrm{GL}(4, \mathbb{Z}_p)$) if $\psi(V)$ is isotropic and $(p^4 - 1)p^3(p^4 - p^2)(p^4 - p^3)$ of them otherwise.

2. Let us consider a non-solvable case. The following is a nice example of Borovic ([Bo]; see also [GW]). Originally it was studied in connection with embeddings of finite groups in the exceptional group $E_8(\mathbb{C})$, but it is also relevant to our case. Let A_6 be the alternating group on 6 letters. It admits a non-trivial triple cover $\widehat{A_6}$, which is unique up to isomorphism. There are 3 irreducible 9-dimensional representations of $\widehat{A_6}$. Exactly one of them, say α, factors through A_6. Thus $\mathrm{Aut}_w(\bar{\alpha}) = \mathrm{Aut}(A_6)$ acts trivially on $\bar{\alpha}$. (Recall that A_6 is special in that $[\mathrm{Aut}(A_6)\colon S_6] = 2$.) Let $\beta_i, i = 1, 2$ be the other 9-dimensional representations. Looking at the character table of $\widehat{A_6}$ we see that $\bar{\beta}_i \sim_w \bar{\alpha}$. Since $\widehat{A_6}$ has no non-trivial characters, $\bar{\beta}_1 \not\sim_s \bar{\beta}_2$. Let ϕ be an outer automorphism of A_6 coming from S_6. We claim that $\bar{\beta}_2 \sim_s \bar{\beta}_1 \circ \phi$. For otherwise $\bar{\beta}_1 \sim_s \bar{\beta}_1 \circ \phi$, giving rise to a projective representation of S_6. However there is no non-trivial triple cover of S_6 ([Su]). We conclude that

$$\mathcal{X}(\bar{\alpha}) = \{\bar{\alpha}, \bar{\beta}_1, \bar{\beta}_2\}$$

with orbits $\{\bar{\alpha}\}, \{\bar{\beta}_1, \bar{\beta}_2\}$. Of course, we can find a Galois representation factoring through α. However, with the present knowledge, this example does not carry over to the automorphic side.

3. We now give an example showing that there is no hope for the method of proof of theorem 2 to work if we only assume that E/F is subsolvable. More specifically, we will give an example of two representations π_1, π_2 of a finite solvable group, such that π_1 is induced from a character of a subnormal subgroup, π_2 is not monomial, but nevertheless their projectivizations are weakly equivalent. Thus, if we realize π_1, π_2 as Galois representations then π_1 is automorphic, but it is not clear that π_2 is automorphic.

   Recall that if
$$1 \longrightarrow N \longrightarrow G \longrightarrow K \longrightarrow 1$$
   is an exact sequence of groups, and π is an irreducible representation of N such that $\pi^g \simeq \pi$ for all $g \in G$, then the obstruction of extending π to G lies in $H^2(K, \mathbb{C}^*)$ and is given by
$$\alpha(x,y) = A(g_x)A(g_y)A(g_{xy})^{-1}\pi(g_{xy}g_y^{-1}g_x^{-1}).$$
   Here, $\{g_x\}_{x\in N}$ is any transversal of N in G and $A(g)$ is an intertwining operator of π and π^g. In this case, $\mathrm{End}_G(\mathrm{Ind}_N^G \pi) \simeq \mathbb{C}[K, \alpha]$

   To construct the sought after example, suppose that we are given two Abelian groups A, B of odd and coprime orders with an action γ of B on A. Suppose that there exist two pairs of non-degenerate $\mathbb{Q}/\mathbb{Z}$-valued alternating bilinear forms $\langle\cdot,\cdot\rangle_i, [\cdot,\cdot]_i, i = 1,2$ on A and B respectively with the following properties:

   (a) B preserves $\langle\cdot,\cdot\rangle_i, i = 1,2$.
   (b) There exist maximal $\langle\cdot,\cdot\rangle_1$ ($[\cdot,\cdot]_1$)-isotropic subgroups A_1 (B_1) of A (B) respectively so that A_1 is B_1-invariant and B_1 acts trivially on A/A_1.
   (c) There do not exist maximal $\langle\cdot,\cdot\rangle_2$ ($[\cdot,\cdot]_2$)-isotropic subgroups A_2 (B_2) of A (B) respectively so that A_2 is B_2-invariant.

   Let $H_i, i = 1,2$ be the Heisenberg group attached to $A, \langle\cdot,\cdot\rangle_i$. This is a central extension of $Z = \mathbb{Q}/\mathbb{Z}$ by A_i with a cocycle corresponding to $\langle\cdot,\cdot\rangle_i$. Let ψ be the character $z \mapsto e^{2\pi i z}$ of Z and let τ_i be the Stone–von Neumann representation of H_i with central character ψ. This is a minimal representation. Let $\mathrm{Aut}_c(H_i)$ be the automorphisms of H_i which act trivially on the center. The exact sequence
$$0 \longrightarrow \mathrm{Inn}(H_i)/Z \longrightarrow \mathrm{Aut}_c(H_i) \longrightarrow \mathrm{Out}_c(H_i) \simeq \mathrm{Sp}(A_i, \langle\cdot,\cdot\rangle_i) \longrightarrow 0$$

splits. Thus, the extensions of H_i by a group K which acts via some $\delta\colon K \to \mathrm{Out}_c(H_i)$ are classified by $H^2(K, \mathbb{Q}/\mathbb{Z})$. Let G_i be the extension of H_i by B defined by the cocycle β_i corresponding to $[\cdot,\cdot]_i$. Note that $W = G_i/Z \simeq A \rtimes_\gamma B$.

PROPOSITION 1: *Under the above assumptions:*

(a) $\mathrm{Ind}_{H_i}^{G_i} \tau_i$ *is isotypic.*

(b) *Its irreducible component* π_i *is a minimal representation.*

(c) *Viewed as projective representations of* W *we have* $\pi_1 \sim_w \pi_2$.

(d) π_1 *is induced from a subnormal subgroup of* G_1.

(e) π_2 *is not monomial.*

Proof:

(a) Clearly $\tau_i^g \simeq \tau_i$ for any $g \in \mathrm{Aut}_c(H_i)$. Since $(|A|, |B|) = 1$, τ_i extends to a representation of $H_i \rtimes B$. Thus, the obstruction of lifting τ_i to G_i is given by β_i. Hence, $\mathrm{End}(\mathrm{Ind}_{H_i}^{G_i} \tau_i)$ is simple and $\mathrm{Ind}_{H_i}^{G_i} \tau_i$ is isotypic.

(b) This follows immediately from 1 and the minimality of τ_i.

(c) This follows from Lemma 2.

(d) By our conditions we can lift the subquotient A_1 to a subgroup of H_1 and B_1 to a subgroup of G_1. The formula $\theta_1(zab) = \psi(z)$ defines a character on the subnormal subgroup $K_1 = ZA_1B_1$, extending ψ. Thus $\pi_1 = \mathrm{Ind}_{K_1}^{G_1} \theta_1$.

(e) Suppose that θ_2 is an extension of ψ to a subgroup K_2 of G_2. After conjugation we can assume that the image of K_2 in W is A_2B_2 with $A_2 < A, B_2 < B$. Then necessarily A_2 is $\langle\cdot,\cdot\rangle_2$-isotropic and B_2 is $[\cdot,\cdot]_2$-isotropic. Since A_2, B_2 cannot be both maximal by our conditions, π_2 is not monomial. ∎

It is east to construct $A, B, \langle\cdot,\cdot\rangle_i, [\cdot,\cdot]_i, i = 1, 2$ as above.

4. Next, let us give an example for which theorem 2 is applicable and for which $\mathcal{X}(\alpha)$ is not homogeneous. Again, this is a minimal projective representation of a meta Abelian group $G = A \rtimes B$. Let $A, \langle\cdot,\cdot\rangle$ be a 4-dimensional simplectic vector space over $\mathbb{Z}_p$ for some $p > 2$ and let B_1 be the unipotent radical of the Siegel parabolic of $\mathrm{Sp}(A)$, acting on A by

α_1. Let H be the Heisenberg group attached to A. Put $B = B_1 \times B_1$, acting on A by $\alpha_1 \times 1$. Note that the center of G is $Z' = A_1 \times (1 \times B_1)$ where A_1 is maximal isotropic. Let now $[\cdot,\cdot]_1$ (resp. $[\cdot,\cdot]_2$) be a non-degenerate skew symmetric form on B for which $1 \times B_1$ is isotropic (non-isotropic). As before, we construct from this data two minimal projective representations π_i of G by inducing the Stone–von Neumann representation τ of H to the extension of H by B defined by $[\cdot,\cdot]_i$. Here we use the fact that τ extends to a representation of $H \rtimes B$. This is true since $\tau = \mathrm{Ind}_{A'}^{H} \theta$ where $A' = ZA_1$ and $\theta(za) = \psi(z)$ is invariant under B (Z, ψ are as before). Thus $\pi_1 \sim_w \pi_2$ and Theorem 2 certainly applies. However π_1 is the projectivization of a representation induced from a character on the inverse image of Z' and this is not true for π_2. Thus π_2 is not obtained from π_1 by an automorphism of G and $\mathcal{X}(\pi_1)$ is non-homogeneous.

5. In the rest of the article, we focus on the case of representations induced from characters on a normal subgroup with a cyclic quotient.

5. Multiplicities of endoscopic L-packets induced from elliptic tori: basic facts

Let us now turn into a special case, namely the simplest endoscopic L-packets — those induced from elliptic tori. Let then E be a cyclic extension of F of degree n, and θ a Hecke character of E. Let $\tilde{\pi} = \tilde{\pi}(\theta)$ be the representation of $\mathrm{GL}(n)$ given by the automorphic induction of θ. The existence of these representations was stated in [K] but was not proved before the general result of [AC]. (In the local case, the lifting was proved by Kazhdan in [K]; cf. [H]. See [HH] for a more complete history). $\tilde{\pi}(\theta)$ will be cuspidal if and only if $\theta^g \neq \theta$ for any $1 \neq g \in \mathcal{G} = \mathrm{Gal}(E/F)$. Henceforth we assume that this is the case. Fix a generator σ of $\mathcal{G}$. $\mathcal{G}$ acts on $W_E^{ab} \simeq C_E$ and on $\widehat{W_E}$, hence also does $\mathbb{Z}[\mathcal{G}]$. It will be convenient to use additive notation, e.g. $(\sigma - 1)\theta$ is the character $a \mapsto \theta(a^\sigma/a)$. Consider the following equivalence relations on characters of C_E:

1. $\theta_1 \sim_s \theta_2$ if there exists $\alpha \in \mathbb{Z}_n$ s.t. $(\sigma - 1)\sigma^\alpha \theta_1 = (\sigma - 1)\theta_2$.

2. $\theta_1 \sim_w \theta_2$ if for every $a \in C_E$ there exists $C \in \mathbb{C}^*$ such that the multi-sets $\{\sigma^i \theta_1(a)\}$ and $C\{\sigma^i \theta_2(a)\}$ are equal.

3. $\theta_1 \sim_{ad} \theta_2$ if $\mathrm{Ad} \circ \oplus \sigma^i \theta_1 \simeq \mathrm{Ad} \circ \oplus \sigma^i \theta_2$ where $\mathrm{Ad}\colon \mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(n^2,\mathbb{C})$ is the adjoint representation.

5.1 REMARKS.

1. $\sim_s \Rightarrow \sim_w \Rightarrow \sim_{ad}$. For $n = 2$ they are all equivalent.

2. If $\theta_1 \sim_s \theta_2$ and $x \in \mathbb{Z}[\mathcal{G}]$ then $x\theta_1 \sim_s x\theta_2$. Similarly for $\sim_w$ and $\sim_{ad}$.

3. For all these relations the class of θ depends only on its restriction to the norm one elements $C_E^1 = (\sigma - 1)C_E$ of C_E.

4. The above equivalence relations on characters reflect the corresponding relations for the Weil group representations induced by them, i.e., $\mathrm{Ind}_{W_E}^{W_F} \theta_1 \sim_s \mathrm{Ind}_{W_E}^{W_F} \theta_2 \iff \theta_1 \sim_s \theta_2$, and similarly for the others. To see this for $\sim_w$ one has to note that for any $a \in W_F$

$$\mathrm{Res}_{\langle a \rangle}^{W_F} \mathrm{Ind}_{W_E}^{W_F} \theta = \bigoplus_{\tau \in W_F / W_E \langle a \rangle} \mathrm{Ind}_{\langle a \rangle \bigcap W_E}^{\langle a \rangle} \theta^\tau.$$

5. From this we can infer that $\tilde{\pi}(\theta_1) \sim_s \tilde{\pi}(\theta_2) \iff \theta_1 \sim_s \theta_2$, and analogously for $\sim_w$. This follows from the basic properties of automorphic induction, strong multiplicity one for $\mathrm{GL}(n)$, and Chebotarev's density theorem.

6. Let
$$x_\alpha = \frac{\sigma^\alpha - 1}{\sigma - 1} \in \mathbb{Z}[\mathcal{G}]/\langle \sum \sigma^i \rangle, \quad \text{for } \alpha \in \mathbb{Z}_n.$$
It is clear that $\theta_1 \sim_{ad} \theta_2$ implies that there exists an α such that $\theta_1 \sim_s x_\alpha \theta_2$.

LEMMA 3: *If $\theta_1 \sim_w \theta_2$ then $n\theta_1 \sim_s n\theta_2$.*

Proof: For each $a \in C_E$ choose a permutation π of $\mathbb{Z}_n$ and $C \in \mathbb{C}^*$ so that $\sigma^i \theta_1(a) = \sigma^{\pi(i)} \theta_2(a) C$. We have $(\sigma^i - 1)\theta_1(a) = (\sigma^{\pi(i)} - \sigma^\beta)\theta_2(a)$, where $\beta = \pi(0)$. Multiplying over i and taking $a = (\sigma - 1)b$ we get $n(\sigma - 1)\theta_1(b) = n(\sigma - 1)\sigma^\beta \theta_2(b)$. Similarly we get

$$n(\sigma - 1)\sigma^i \theta_1(b) = n(\sigma - 1)\sigma^{\pi(i)} \theta_2(b) \tag{6}$$

for any i. A priori π depends on b but now we can choose a permutation so that (6) holds for all $b \in A$. ∎

Let us define G_θ to be the equivalence classes of $\{x \in \mathbb{Z}[\mathcal{G}]: x\theta \sim_w \theta\}$ under the relation $x\theta \sim_s y\theta$. By Remark 2 above, the product on G_θ makes sense.

THEOREM 3: *Let θ be a Hecke character of E. Then G_θ forms an Abelian group, $\mathcal{X}(\tilde{\pi}(\theta)) \simeq G_\theta$, and $\mathcal{M}(\mathcal{L}(\tilde{\pi}(\theta))) = |G_\theta| < n$.*

Proof: It follows from Remarks 6 and 2 that G_θ is a group of order $< n$. Let ω be a character of $\mathbb{I}_F$ with $\operatorname{Ker}\omega = F^* \operatorname{Nm} \mathbb{I}_E^*$. The cuspidal representations $\tilde{\pi}$ which are automorphic induction from characters of E are characterized by the property that $\tilde{\pi} \otimes \omega \simeq \tilde{\pi}$, and thus they are stable under $\sim_w$. Let $\alpha = \operatorname{Ind}_{W_E}^{W_F} \theta$: $W_F \longrightarrow \mathrm{PGL}(n, \mathbb{C})$ and $A = \operatorname{Im}\alpha$. We can assume that $\alpha(W_E)$ is diagonal. Choose $\bar{\sigma} \in W_F$ above σ. If $\theta_1 \sim_w \theta$ we can construct an element in $\operatorname{Aut}_w(A)$ sending the diagonal matrix $(\theta(a^g))_{g\in\mathcal{G}}$ to $(\theta_1(a^g))_{g\in\mathcal{G}}$ for $a \in W_E$, and $\alpha(\bar{\sigma})$ to itself. This gives an isomorphism of G_θ with $\mathcal{X}(\tilde{\pi}(\theta))$. It remains to invoke 5. ∎

Remark:

1. We could of course appeal to Theorem 2 to conclude that $\mathcal{M}(\mathcal{L}(\tilde{\pi}(\theta))) = |\mathcal{X}(\tilde{\pi}(\theta))|$, but this case is easy to analyze directly.

2. The above characterization of cuspidal representations induced from characters (in the cyclic case) in terms of being self twists under a Hecke character is proved in [AC] only in the prime case. This is mainly for historical reasons. In any case Labesse ([Lab]) treats the general case.

6. Multiplicities of endoscopic L-packets of SL(n), n prime

We keep the same notations as in the previous section but in this section we will assume that $n > 2$ is prime. Let ϵ: $\mathbb{Z}[\mathcal{G}] \longrightarrow \mathbb{Z}$ be the augmentation homomorphism.

LEMMA 4: *Let $\tau \in \mathbb{Z}[\mathcal{G}]$ and suppose that $(\epsilon(\tau), n) = 1$. Then there exists $\psi \in \mathbb{Z}[\mathcal{G}]$ such that $\psi\tau = m$ for some $m \in \mathbb{Z}$ with $(m, n) = 1$.*

Proof: If $\tau = \sum_{i=0}^{n-1} a_i \sigma^i$ then the equation $\psi\tau = 1$ is a linear system whose coefficients matrix is $\{a_{i-j}\}_{i,j=0,\dots,n-1}$. This matrix has determinant $\prod_{i=0}^{n-1} \sum_{j=0}^{n-1} a_j \zeta^{ij}$ where ζ is a primitive n-th root of unity. This product is congruent to $\epsilon(\tau)^n \bmod (\zeta - 1)$. Since $(\zeta - 1)^{\varphi(n)} = (n)$ in $\mathbb{Z}[\zeta]$ the Lemma follows from Cramer's rule. ∎

We say that a character is **torsion** (resp. n**-torsion**, $\tilde{n}$**-torsion**) if its order is finite (resp. an n-power, relatively prime to n).

THEOREM 4: *If $G_\theta \neq 1$ then $\theta\big|_{C_E^1}$ is torsion, but not $\tilde{n}$-torsion and ϵ induces an injective homomorphism of G_θ into $\mathbb{Z}_n^*$.*

Proof: Suppose that $G_\theta \neq 1$. By 6, $x_\alpha\theta \sim_w \theta$ but $x_\alpha\theta \not\sim_s \theta$ for some α. By Lemma 3, $n(x_\alpha - \sigma^i)(\sigma - 1)\theta = 0$ for some i and by Lemma 4, $(\sigma-1)\theta$ is torsion. On the other hand, it follows from Lemma 3 that $(\sigma-1)\theta$ is not $\tilde{n}$-torsion, since $x_\alpha\theta \not\sim_s \theta$. Now, $x\theta \sim_s \theta$ for $x \in \mathbb{Z}[\mathcal{G}]$ means that $(x - \sigma^i)(\sigma-1)\theta = 0$ for some i, and by Lemma 4 we get that $\epsilon(x) = 1$. On the other hand if $x\theta \sim_w \theta$ with $\epsilon(x) = 1$ then by Remark 6 of Section 5.1 we get that $x\theta \sim_s x_\alpha\theta$ for $\alpha \in \mathbb{Z}_n$, and again by Lemma 4, $\alpha = 1$, otherwise $(\sigma - 1)\theta$ would be $\tilde{n}$-torsion. ∎

Now we want to have some more information about the group G_θ.

6.1 THE n-TORSION CASE.

PROPOSITION 2: *Suppose that $\theta\big|_{C_E^1}$ is n-torsion and $G_\theta \neq 1$. Then $\theta\big|_{C_E^1}$ has order n and one of the following holds:*

1. *$(\sigma-1)^2\theta = 0$ and then $G_\theta = \mathbb{Z}_n^*$ or,*
2. *1 does not hold but $(\sigma-1)^3\theta = 0$. Then $G_\theta = \{$quadratic residues $\bmod n\}$, or*
3. *1 and 2 do not hold but $(\sigma-1)^4\theta = 0$ and $n \equiv 1 \pmod 4$, in which case $G_\theta = \{\pm 1\}$.*

Proof: Suppose $1 \neq \alpha \in G_\theta$. By Lemma 3 we infer that $n(\sigma-1)(x_\alpha - \sigma^\beta)\theta = 0$, for some β. Using Lemma 4 once again we conclude that $n(\sigma-1)\theta = 0$.

Suppose that 1 holds. For any i, $(\sigma^i - 1)\theta = (\sigma-1)x_i\theta = (\sigma-1)i\theta$. The same is true for $x_\alpha\theta$ for any $\alpha \in \mathbb{Z}_n^*$ so that

$$\{\sigma^i\theta(a)\} = \theta(a)\{(\sigma^i - 1)\theta(a)\} = \theta(a)\{i(\sigma-1)\theta(a)\}$$

and

$$\{\sigma^i x_\alpha\theta(a)\} = x_\alpha\theta(a)\{i(\sigma-1)x_\alpha\theta(a)\} = x_\alpha\theta(a)\{i\alpha(\sigma-1)\theta(a)\}$$

and the first part follows since $(\sigma-1)\theta(a)$ is an n-th root of unity.

Suppose now that 2 is satisfied. Let $s = (\sigma-1)\theta(a)$, $t = (\sigma-1)^2\theta(a)$. Then we have $(\sigma-1)\sigma^i\theta(a) = ((\sigma-1) + x_i(\sigma-1)^2)\theta(a) = st^i$, so that $(\sigma^i - 1)\theta(a) = \sum_{j=0}^{i-1}(\sigma-1)\sigma^j\theta(a) = s^i t^{\binom{i}{2}}$. We have an analogous formula for $x_\alpha\theta$ with $s' = (\sigma^\alpha - 1)\theta(a) = s^\alpha t^{\binom{\alpha}{2}}$ and $t' = t^\alpha$. Choosing $t \neq 1$ and rewrite the condition

$$\{(\sigma^i - 1)\theta(a)\} = \text{const}\{(\sigma^i - 1)x_\alpha\theta\}$$

as a congruence relation modulo n. (Note that the implied constant is an n-th root of unity.) We get two polynomials in $\mathbb{Z}_n$ of degree 2 whose leading terms differ by a factor of α and whose images differ by an additive constant (as multi-sets). Since we can transfer the polynomials into monomials of degree 2 without changing this property, it is clear that this can hold if and only if α is a quadratic residue mod n.

Suppose now that 1 does not hold. Since $x_\alpha\theta \sim_{ad} \theta$ we know that for any i, $(x_\alpha(\sigma^i - 1) - \sigma^\beta(\sigma^j - 1))\theta = 0$ for some β, where necessarily (by Lemma 4) $j = i\alpha$. Taking $i = 2$ we find that $(\sigma^\alpha - 1)(\sigma + 1 - \sigma^\beta(\sigma^\alpha + 1))\theta = 0$, or $(\sigma^\alpha - 1)(\sigma - 1)(\sigma x_{\beta-1} + x_{\alpha+\beta})\theta = 0$. Once again, by Lemma 4 we can infer that $1 - \beta = \alpha + \beta$. If $\alpha \neq -1$ it means that $(\sigma - 1)^3\theta = 0$.

It remains therefore to consider the case $\alpha = -1$. Certainly $x_\alpha\theta \sim_s -\theta$. Since $n(\sigma - 1)\theta = 0$ it follows that $(\sigma - 1)^n\theta = 0$ (in fact the two conditions are equivalent). For any $a \in C_E$, let $\xi_i(a) = (\sigma - 1)^i\theta(a), i = 0, 1, \ldots, n-1$. Let $b = \sum_{i=0}^{n-1} k_i\sigma^i a$. A straightforward computation, generalizing the previous ones, yields

$$\begin{aligned}(\xi_{n-1}(b), \ldots, \xi_0(b)) = \\ &(\xi_{n-1}(a)^{\sum k_i}, \xi_{n-2}(a)^{\sum k_i}\xi_{n-1}(a)^{\sum ik_i}, \ldots, \\ &\xi_0(a)^{\sum k_i}\xi_1(a)^{\sum ik_i}\xi_2(a)^{\sum \binom{i}{2}k_i} \cdot \xi_{n-1}(a)^{\sum \binom{i}{n-1}k_i}).\end{aligned}$$

Let $k \le n$ be minimal such that $(\sigma-1)^k\theta = 0$ and suppose that $(\sigma-1)^{k-1}\theta(a) \neq 1$. From the computation it follows easily that every k-tuple $(\zeta^{\eta_0}, \ldots, \zeta^{\eta_{k-1}})$ of n-th roots of unity can be represented as $(\xi_0(b), \ldots, \xi_{k-1}(b))$ for some b of the above form. On the other hand, by the same computation $\sigma^i\theta(b) = \zeta^{\sum_{j=0}^{k-1}\binom{i}{j}\eta_j}$. The exponent can be an arbitrary polynomial (in i) of degree $\le k$. We are reduced to the question of whether for an arbitrary polynomial $f(x)$ over $\mathbb{Z}_n$ of degree $\le k$, there exists c so that

$$\{f(i)\} = \{-f(i)\} + c \tag{7}$$

as multi-sets. For f of degree 2 we saw that this holds if and only if -1 is a square. For f of degree 1 or 3 this clearly holds automatically if $n > 3$. However, this condition cannot hold for $k = 4$. For example, take $f(x) = x^2(x^2 - 1)$ and suppose that (7) holds. Since the value 0 is obtained thrice and all other values are obtained an even number of times, $c = 0$. We could infer that $\sum_{i\in\mathbb{Z}_n} f(i)^l = 0$ whenever l is odd. We can write this as $\sum_{j=0}^{l}(-1)^{l-j}\binom{l}{j}\delta_{2j+2l}$ where $\delta_m = \sum_{i\in\mathbb{Z}_n} i^m = -1$ if $m = n-1$ and 0 otherwise. By choosing l to be the smallest odd integer $\ge (n-1)/4$ we get a contradiction. These considerations complete the proof of the Proposition. ∎

Remark: In the same way as in part 1 of the Proposition it can be shown that for general n and a character θ with $(\sigma-1)^2\theta = 0$, we have $G_\theta \simeq \mathbb{Z}_n^*$. The examples in [B] are of this type.

6.2 CONDITIONS ON THE $\tilde{n}$-TORSION PART. Recall that we want to classify the condition $G_\theta \neq 1$. We already know what happens in the cases where $\theta|_{C_E^1}$ is either n-torsion or $\tilde{n}$-torsion. To treat the general case let $1 \neq \alpha \in \mathbb{Z}_n^*$ and let η,ψ be the restrictions of θ to the n-torsion and $\tilde{n}$-torsion parts $A_n, A_{\tilde{n}}$ of $(\sigma-1)\theta$ respectively. We shall assume that $(\sigma-1)\eta, (\sigma-1)\psi \neq 0$.

PROPOSITION 3: *$\alpha \in G_\theta$ if an only if $(\sigma-1)^2\eta = 0$ and for any $b \in (\sigma-1)A_{\tilde{n}}$ there exists t so that*

$$\sigma^{i\alpha}\psi(b) = \sigma^{t+i}\psi(b) \quad \text{for all } i. \tag{8}$$

Proof: Suppose that $\alpha \in G_\theta$. Arguing as before with Lemmas 3 and 4 we get that $n(\sigma-1)\eta = 0$ and there exists β such that

$$\sigma^\beta(\sigma-1)\psi = (\sigma-1)x_\alpha\psi. \tag{9}$$

Choose $a \in A_n$ such that $\zeta = (\sigma-1)\eta(a) \neq 1$, but $(\sigma-1)^i\eta(a) = 1$ for any $i > 1$. Take any $b \in A_{\tilde{n}}$ and write $\xi_i = (\sigma^i - 1)\psi(b)$. By our assumption we have

$$\{(\sigma^i-1)\eta(a)(\sigma^i-1)\psi(b))\} = \mathrm{const}\{(\sigma^i-1)x_\alpha\eta(a)(\sigma^i-1)\sigma^\beta\psi(b)\} \tag{10}$$

and according to the computations of Proposition 2 this means that $\{\zeta^i\xi_i\} = \mathrm{const}\{\zeta^{i\alpha}\xi_i\}$. The implied constant is certainly an n-th root of unity so write it as ζ^t. Let π be a permutation achieving this equality. The condition $\zeta^i\xi_i = \zeta^t\zeta^{\pi(i)\alpha}\xi_{\pi(i)}$ implies that $\xi_i = \xi_{\pi(i)}$ and $i = t + \pi(i)\alpha$, so that condition (8) is satisfied for all $b \in A_{\tilde{n}}$.

In the converse direction, one can reverse the arguments to conclude that $\alpha \in G_\theta$ as long as $(\sigma-1)^2\eta = 0$ and (8) is satisfied for all $b \in A_{\tilde{n}}$. To prove the last statement, note first that (8) implies that $\oplus\sigma^i(\sigma-1)(\sigma^\alpha-1)\psi = \oplus\sigma^i(\sigma-1)^2\psi$. Hence, $(\sigma-1)^2(x_\alpha - \sigma^\beta)\psi = 0$ for some β. However, $(\sigma-1)^2 | n(\sigma-1)$ in $\mathbb{Z}[\mathcal{G}]$, so that (9) is satisfied. Now, by (9) and (8) applied to $(\sigma-1)b$ we infer that $\sigma^{i\alpha}(\sigma^\alpha-1)\psi(b) = \sigma^{i+t+\beta}(\sigma-1)\psi(b)$ for some t. This means that $c = (\sigma^{i\alpha} - \sigma^{i+t+\beta})\psi(b)$ does not depend on i. However, $c^n = 1$, whence $c = 1$ and (8) is indeed satisfied for all $b \in A_{\tilde{n}}$.

It remains to show that $\alpha \in G_\theta$ implies that $(\sigma-1)^2\eta = 0$. Assume on the contrary that $(\sigma-1)^2\eta \neq 0$. The same arguments as in Proposition 2 yield that either $(\sigma-1)^3\theta = 0$ or $\alpha = -1$ and $n \equiv 1 \pmod 4$. The first alternative implies that $n(\sigma-1)\theta=0$ which contradicts our assumption. Suppose then that $\alpha=-1$. In condition (9), $\beta=-1$, i.e., $2(\sigma-1)\psi=0$, because otherwise $(1+\sigma^{\beta+1})(\sigma-1)\psi=0$ which would imply that $(\sigma-1)\psi=0$ since $(1+\sigma^{\beta+1})=(\sigma^{2(\beta+1)}-1)/(\sigma^{\beta+1}-1)$ is invertible. We already know that (8) is satisfied. For any k we can take an element $a\in A_n$ such that $\zeta=(\sigma-1)^2\eta(a)\neq 1,(\sigma-1)\eta(a)=\zeta^k$, but $(\sigma-1)^i\eta(a)=1$ for any $i>2$. Writing the condition (10) using the computation of Proposition 2 yields $\{\zeta^{\binom{i}{2}+ki}\xi_i\}=c\{\zeta^{-\binom{i}{2}-ki}\xi_i\}$. Again, $c=\zeta^t$ and if π implements this equality it is easy to see that $\xi_i=\xi_{\pi(i)}$ and $\{\pi(s+i),\pi(s-i)\}=\{s+i\gamma,s-i\gamma\}$, for $s=1/2-k$, $\gamma^2=-1$ and any i. Taking s so that $\xi_{s+i}=\xi_{s-i}$ for each i (by (8)), we see that (8) is satisfied for $\alpha=\gamma$ as well. This will contradict the next Proposition. ∎

PROPOSITION 4: *Let ψ be as above and $B=A_{\tilde n}$. Then ψ satisfies the condition (8) for any $b\in(\sigma-1)B$ if and only if there exist primes q,r such that*

$$n=\frac{q^{r^s}-1}{q^{r^{s-1}}-1},\quad q(\sigma-1)\psi=0,\quad (\sigma-1)g(\sigma)\psi=0$$

for some irreducible polynomial $g(x)\in\mathbb{Z}_q[x]$ which divides $(x^n-1)/(x-1)$ and $\alpha\in\langle q^{r^{s-1}}\rangle$. In particular α has a prime order.

Proof: Suppose first that $q(\sigma-1)\psi=0$ for some prime q. The set of all n-tuples of the form $\{(\psi(b),\sigma\psi(b),\dots,\sigma^{n-1}\psi(b))\colon b\in(\sigma-1)B\}$ can be thought of as an ideal I in $\mathbb{Z}_q[x]/(x^n-1)$ and is therefore generated by a polynomial $f(x)|x^n-1$. Certainly $x-1|f(x)$. In this setting condition (8) translates into the following:

$$\text{for any } h(x)\in I \text{ there exists } k \text{ so that}$$
$$\phi(x)=x^kh(x) \text{ satisfies } \phi(x^\alpha)\equiv\phi(x) \tag{11}$$

(unless otherwise indicated $\equiv$ will always mean $(\operatorname{mod} x^n-1)$). Moreover, it is clear that this k is unique unless $h(x)\equiv 0$ (since $x-1\in I$). In particular suppose that

$$f(x^\alpha)\equiv x^\lambda f(x). \tag{12}$$

Let $g(x)=(x^n-1)/f(x)$ and let $H=\{h(x)\in\mathbb{Z}_q[x]/(g(x))\colon h(x^\alpha)\equiv h(x) \pmod{g(x)}\}$. This is well defined by (12). Let $d=\dim H$. Clearly $h(x)\in H$ if and only if $\phi(x)=f(x)h(x)$ satisfies $\phi(x^\alpha)\equiv x^\lambda\phi(x)$. This and (11) imply that $\bigcup_{k\in\mathbb{Z}_n}x^kH=\mathbb{Z}_q[x]/(g(x))$, where the union is disjoint, except for 0. We get $n(q^d-1)=q^m-1$ where $m=\deg(g)$. Clearly, this implies that $m=r^s$ and $d=r^{s-1}$ for some r prime. Also, the order of q in $\mathbb{Z}_n^*$ is m. Hence, $g(x)$ is irreducible over $\mathbb{Z}_q$. This and (12) imply that $\alpha\in\langle q\rangle$. H can now be viewed as the subfield of $\mathrm{GF}(q^m)$ of the invariant elements under the transformation $x \mapsto x^\alpha$, and thus has dimension $m/|\alpha|$. Hence α has order r. It is clear that $(\sigma-1)g(\sigma)\psi = 0$.

The arguments can be reversed provided that (12) is satisfied. To see this, take a root y of $g(x)$ and note that $f(y)^{\alpha-1}$ is of order n and hence can be expressed as y^λ.

It remains to prove that condition (8) implies that $q(\sigma-1)\psi = 0$. It is already clear that $(\sigma-1)\psi$ is q-torsion where q is determined by α (α represented in $\{0,\dots,n-1\}$ is a q-power). Suppose on the contrary that $q(\sigma-1)\psi \neq 0$. We can assume that $q^2(\sigma-1)\psi = 0$. Again we have the set $\{(\psi(b),\dots,\sigma^{n-1}\psi(b)), b \in (\sigma-1)B\}$ which can be thought of as an ideal I in $\mathbb{Z}_{q^2}[x]/(x^n-1)$. This ring is not a principal ideal domain any more. However property (11) still holds. Let $\phi\colon \mathbb{Z}_{q^2}[x]/(x^n-1) \longrightarrow \mathbb{Z}_q[x]/(x^n-1)$ be the canonical homomorphism, and let $f(x) \in I$ be such that $qf(x) \not\equiv 0$. It is clear that the $\phi(I)$ is the corresponding module for $q\psi$ and hence by the first part of the proof we know that $\mathbb{Z}_q[x]/\{h(x)\colon h(x)\phi(I) \equiv 0\}$ has cardinality q^{r^s} and is an irreducible module over $\mathbb{Z}_q[x]/(x^n-1)$. Hence it is equal to $\mathbb{Z}_q[x]/\{h(x)\colon h(x)\phi(f(x)) \equiv 0\}$. It is still true as before that $\bigcup_{k\in\mathbb{Z}_n} x^k H = M$, where now $H = \{h(x)\colon f(x)h(x^\alpha) \equiv f(x)h(x)\}/\{h(x)\colon f(x)h(x) \equiv 0\}$ and $M = \mathbb{Z}_{q^2}[x]/\{h(x)\colon f(x)h(x) \equiv 0\}$. Moreover, the union is disjoint except for 0. Again, this implies that $n = (q^m-1)/(q^d-1)$ where $|H| = q^d$, $|M| = q^m$. Hence $m = r^s$. However this would imply that the canonical surjection $M \longrightarrow \mathbb{Z}_q[x]/\{h(x)\colon h(x)\phi(f(x)) \equiv 0\}$ is a bijection and this is absurd since q is in the kernel. This finishes the proof of the Proposition. ∎

6.3 Final classification.

THEOREM 5: *Let $F \subset E$ be a cyclic extension of prime order n and let θ be a Hecke character of E. Denote by θ' its (non-trivial) restriction to the norm one elements. Let $G_\theta \subset \mathbb{Z}_n^*$ be as above. Then $\mathcal{M}(\mathcal{L}(\tilde{\pi}(\theta))) = |G_\theta|$. Moreover $G_\theta \neq 1$ if and only if one of the following happens:*

1. *θ' is $\mathcal{G}$-invariant.*
2. *$(\sigma-1)^2\theta' = 0$ (but not 5).*
3. *$n \equiv 1 \pmod 4$ and $(\sigma-1)^3\theta' = 0$ (but not 5 or 5).*
4. *$n = (q^{r^s}-1)/(q^{r^{s-1}}-1)$ for some primes q, r and some s, $q(\sigma-1)\theta' = 0$ (but $q\theta' \neq 0$) and $ng(\sigma)\theta' = 0$ (but $n\theta' \neq 0$) for some irreducible polynomial $g(x) \in \mathbb{Z}_q[x]$ (of degree r^s) which divides $(x^n-1)/(x-1)$.*

Correspondingly:

1. $G_\theta = \mathbb{Z}_n^*$.
2. $G_\theta = \{$*quadratic residues in* $\mathbb{Z}_n^*\}$.
3. $G_\theta = \pm 1$.
4. $G_\theta = \langle q^{r^{s-1}} \rangle$ *of (prime) order* r.

Moreover, for any appropriate n Hecke characters with the corresponding condition exist.

Proof: This follows from Propositions 2, 3 and 4 along with Theorem 4. The last assertion in the Theorem follows from the fact that any finite $\mathbb{Z}[\mathcal{G}]$-module can be realized as a quotient of C_E. ∎

7. Multiplicities of L-packets induced from elliptic tori: the general case

What happens for general n? Suppose first that $n = p^k$ is a prime power. Let $\sqrt{I}$ denote the radical of an ideal I.

LEMMA 5:

1. $\sqrt{(p(\sigma-1))} = \sqrt{(\sigma-1)} = \epsilon^{-1}((p))$ *in* $\mathbb{Z}_{p^m}[\mathbb{Z}_n]$ *for any* m.
2. *If* $\epsilon(x) \equiv 1 \bmod p$ *then* $x^{p^r} \to 1$ *in the* (p)*-adic topology.*

Proof:

1. Immediate since $(\sigma-1)^n \in (p(\sigma-1))$ and $(\sigma-1) = \epsilon^{-1}(0)$.
2. Follows from $x - 1 \in \sqrt{(p(\sigma-1))}$ in $\mathbb{Z}_{p^m}[\mathbb{Z}_n]$. ∎

PROPOSITION 5: *If n is a prime power $|G_\theta|\,|\varphi(n)$.*

Proof: Indeed, by Lemma 3 we may assume that $(\sigma-1)\theta$ is not $\tilde{p}$-torsion. Also, Lemma 4 and its proof are still valid. We therefore see, by the same argument as in Theorem 4 that ϵ induces a homomorphism $\bar{\epsilon}$: $G_\theta \mapsto \mathbb{Z}_p^*$. Let x be in its kernel. For every s we get by part 2 of Lemma 5 that

$$(\sigma-1)x^{p^{r+k}}\theta = (\sigma-1)\theta + (\sigma-1)p^s y_s \theta \tag{13}$$

for some r and $y_s \in \mathbb{Z}[\mathcal{G}]$. On the other hand $x^{p^r}\theta \sim_w \theta$ so that again by Lemma 3, $n(\sigma-1)x^{p^r}\theta = n(\sigma-1)\sigma^i\theta$ for some i. We conclude that

$$n^n(\sigma-1)x^{p^{r+k}}\theta = n^n(\sigma-1)\theta. \tag{14}$$

From (13) and (14) it follows that $x^{p^{r+k}}\theta \sim_s \theta$. Indeed, $n^n p^s(\sigma - 1)y_s\theta = 0$. However $(\sigma - 1)y_s\theta$ all belong to a finitely generated group of characters, and cannot have arbitrarily large orders, and thus $p^s(\sigma-1)y_s\theta = 0$ for some s. Thus $\operatorname{Ker}\bar{\epsilon}$ is a p-group and since $|G_\theta| < n$ we get the required. ∎

It is natural to ask whether Proposition 5 remains true without the assumption on n. However, this is not true in general as the following example shows.

Let $n = 3q$ for prime q, H an elementary Abelian finite q-group, $\{\chi_i\}_{i=0,1,2}$ characters on H, such that no one is the power of the other but $\chi_0+\chi_1+\chi_2 = 0$. Let $A = H \times \ldots \times H$ (n times) and ν be the character of A with components $0, 0, 0, \chi_0, \chi_1, \chi_2, \ldots, (q-1)\chi_0, (q-1)\chi_1, (q-1)\chi_2$. Realize A as a quotient $\mathbb{Z}[\mathcal{G}]$-module of C_E with σ acting as a cyclic shift. Thus we obtain a Hecke character θ. In this case $G_\theta \simeq \mathbb{Z}_q$. To see this, we have to know for which $\alpha \in \mathbb{Z}_n$ $x_\alpha\theta \sim_w \theta$. Let $\alpha = 3\beta + 1$ and $a \in A$. Clearly, $(\sigma^i\nu(a))_{i=0,\ldots,n-1} = (\phi_i\zeta_i^j)_{i=0,1,2,j=0,\ldots,q-1}$ for some $(\phi_i)_{i=0,1,2}$ and $(\zeta_i)_{i=0,1,2}$ with $\zeta_0\zeta_1\zeta_2 = 1$. Thus, for $\nu' = x_\alpha\nu$, $(\sigma^i\nu'(a))_{i=0,\ldots,n-1} = (\phi'_i\zeta_i^j)_{i=0,1,2,j=0,\ldots,q-1}$, with $\phi'_i = \eta_i\phi_i$ and $\eta_i/\eta_{i-1} = \zeta_i^\beta, i = 1, 2$. Since $\zeta_0\zeta_1\zeta_2 = 1$ either at most one of them is 1 or all are 1. In the first case it is evident that $\{\sigma^i\nu(a)\}_{i=0,\ldots,n-1} = const\{\sigma^i\nu'(a)\}_{i=0,\ldots,n-1}$ as multi-sets. In the latter case η_i does not depend on i, and we get the same. Thus, $x_{3\beta+1} \in G_\theta$. It is easily seen that $x_{3\beta+1} \not\sim_s x_{3\gamma+1}\theta$ when $\beta \not\equiv \gamma \bmod q$. On the other hand if $\alpha = 3\beta$ or $\alpha = 3\beta + 2$ then $x_\alpha\theta \notin G_\theta$, because $x_\alpha\theta \not\sim_{ad} \theta$. For example $(\sigma^3 - 1)x_\alpha\nu$ equals 0 in the first case and

$$(\beta(\chi_0+\chi_1), \beta(\chi_1+\chi_2), \beta(\chi_2+\chi_0), \beta(\chi_0+\chi_1), \ldots)$$

in the latter, and neither can be the same as $(\sigma^i - \sigma^j)\nu$ for any $i \neq j$.

What remains true for general n is that $\mathcal{M}(\mathcal{L}(\tilde{\pi}(\theta)))$ depends only on $\operatorname{Ann}((\sigma-1)\theta)$ (this is not true for the ordinary multiplicities; cf. [B]).

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